SUPPLEMENTARY MATERIALS: ALPERT MULTIWAVELETS AND LEGENDRE-ANGELESCO MULTIPLE ORTHOGONAL POLYNOMIALS*

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S1. Explicit matrices for the scaling relation and the multiwavelets. The Legendre polynomials are denoted by $P_n(x)$ and they are orthogonal on $[-1,1]$

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{n,m}.$$

We will be using the orthonormal Legendre polynomials on $[0,1]$ given by

$$\ell_n(x) = \sqrt{2n+1}P_n(2x-1).$$

We denote the scaling vector by

$$\Phi_n(x) = \begin{pmatrix} \ell_0(x) \\ \ell_1(x) \\ \vdots \\ \ell_{n-1}(x) \end{pmatrix} \chi_{[0,1]}$$

and the vector containing the multiwavelets by

$$\Psi(t) = \begin{pmatrix} f_1^n(2t-1) \\ f_2^n(2t-1) \\ \vdots \\ f_n^n(2t-1) \end{pmatrix} \chi_{[0,1]},$$

where χ_A is the indicator function for the set A. The scaling relation is

$$\Phi_n(\frac{t}{2}) = C_{-1}^n\Phi_n(t) + C_1^n\Phi_n(t-1), \tag{S1}$$

where C_{-1}^n and C_1^n are $n\times n$ matrices (which are lower triangular). The wavelets are expressed in terms of Legendre polynomials by

$$\Psi_n(\frac{t+1}{2}) = D_{-1}^n\Phi_n(t+1) + D_1^n\Phi_n(t), \tag{S2}$$

where D_{-1}^n and D_1^n are $n\times n$ matrices. The orthogonality of the Legendre polynomials gives

$$\int_{-1}^{1} \Phi_n(\frac{x+1}{2})\Phi_n^T(\frac{x+1}{2})\,dx = 2\mathbb{I}_n,$$

where $\mathbb{I}_n$ is the identity matrix of order n, so that

$$C_{-1}^n(C_{-1}^n)^T + C_1^n(C_1^n)^T = 2\mathbb{I}_n. \tag{S3}$$

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The orthogonality of Alpert's multiwavelets gives

$$\int_{-1}^{1} \Psi_n(\frac{x+1}{2})\Psi_n^T(\frac{x+1}{2})\, dx = \mathbb{I}_n,$$

so that

$$D_{-1}^n (D_{-1}^n)^T + D_1^n (D_1^n)^T = \mathbb{I}_n.$$

Furthermore, the multiwavelets in Ψ_n are orthogonal to polynomials of degree $\leq n$, hence

$$\int_{-1}^{1} \Psi_n(\frac{x+1}{2})\Phi_n^T(\frac{x+1}{2})\, dx = \mathbb{O}_n,$$

so that

$$D_{-1}^n (C_{-1}^n)^T + D_1^n (C_1^n)^T = \mathbb{O}_n.$$

S1.1. The scaling relation and the matrices C_{-1}^n. The matrices C_1^n for $1 \leq n \leq 4$ are given explicitly in [S3, p. 2486] (but their notation is a bit different for the multiplicity index n). Here is a simple way to generate them. The orthonormal Legendre polynomials on $[0,1]$ satisfy the three term recurrence relation

$$t\ell_{n-1}(x) = a_n \ell_n(t) + \frac{1}{2}\ell_{n-1}(t) + a_{n-1}\ell_{n-2}(t),$$

where

$$a_n = \frac{n}{2\sqrt{(2n-1)(2n+1)}}.$$

Introducing the Jacobi matrix

$$J_n = \begin{pmatrix} 1/2 & a_1 & 0 & 0 & \cdots & 0 \\ a_1 & 1/2 & a_2 & 0 & \cdots & 0 \\ 0 & a_2 & 1/2 & a_3 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{n-2} & 1/2 & a_{n-1} \\ 0 & \cdots & 0 & 0 & a_{n-1} & 1/2 \end{pmatrix} \tag{S4}$$

then gives

$$J_n \Phi_n(t) = t\Phi_n(t) - a_n \ell_n(t) e_n \chi_{[0,1]}(t), \tag{S5}$$

where e_n is the unit vector $(0,\ldots,0,1)^T$ in $\mathbb{R}^n$. If we use (S5) in the scaling relation (S1), then

$$J_n\Phi_n(t/2) = J_n C_{-1}^n \Phi_n(t) + J_n C_1^n \Phi_n(t-1).$$

Note that J_n is the truncated Jacobi matrix for the Legendre polynomials on $[0,1]$ and that its eigenvalues are the zeros of p_n, which are all in $(0,1)$, hence J_n is not singular (and positive definite), so that its inverse exists. This allows to write

$$J_n\Phi_n(t/2) = J_n C_{-1}^n J_n^{-1} J_n \Phi_n(t) + J_n C_1^n J_n^{-1} J_n \Phi_n(t-1).$$

If we use (S5) three times, then

$$\begin{aligned} \frac{t}{2}\Phi_n(\frac{t}{2}) - a_n\ell_n(\frac{t}{2})\chi_{[0,2]}e_n &= J_n C_{-1}^n J_n^{-1}\Big(t\Phi_n(t) - a_n\ell_n(t)\chi_{[0,1]}e_n\Big) \\ &+ \ J_n C_1^n J_n^{-1}\Big((t-1)\Phi_n(t-1) - a_n\ell_n(t-1)\chi_{[1,2]}e_n\Big). \end{aligned} \tag{S6}$$

Taking $t = 0$ gives

$$-a_n \ell_n(0) e_n = J_n C^n_{-1} J_n^{-1} \Big(-a_n \ell_n(0) e_n \Big),$$

and since $\ell_n(0) \neq 0$ we thus find

$$J_n C^n_{-1} J_n^{-1} e_n = e_n. \tag{S7}$$

Using (S1) on the left side of (S6) and (S7) on the right side gives for $t \in [0, 1]$

$$\frac{t}{2} C^n_{-1} \Phi_n(t) = a_n \ell_n(t/2) e_n - a_n \ell_n(t) e_n + t J_n C^n_{-1} J_n^{-1} \Phi_n(t).$$

Divide this by t, multiply on the right by $\Phi_n^T(t)$ and integrate over $[0, 1]$, to find

$$C^n_{-1} = 2 J_n C^n_{-1} J_n^{-1} + 2a_n \int_0^1 \frac{\ell_n(t/2) - \ell_n(t)}{t} e_n \Phi_n^T(t) \, dt,$$

where we used the orthonormality of the Legendre polynomials

$$\int_0^1 \Phi_n(t) \Phi_n^T(t) \, dt = \mathbb{I}_n.$$

Let us introduce the column vector m_n, with

$$(m_n)_j = 2a_n \int_0^1 \frac{\ell_n(t/2) - \ell_n(t)}{t} p_j(t) \, dt, \qquad 0 \leq j \leq n - 1.$$

and $M_n = e_n m_n^T$ as the matrix with zeros everywhere, except for the last row which is m_n^T, then we have the matrix relation

$$C^n_{-1} = 2 J_n C^n_{-1} J_n^{-1} + M_n,$$

or, after multiplication the the right by J_n,

$$C^n_{-1} J_n = 2 J_n C^n_{-1} + M_n J_n. \tag{S8}$$

This is the matrix version of the scaling relation (S1), and the 2 results from the fact that our scaling is with a factor 2.

We now describe a recursive way to compute the matrices C^n_{-1}. Write

$$C^{n+1}_{-1} = \begin{pmatrix} C^n_{-1} & 0_n \\ r_n^T & s_{n+1} \end{pmatrix}, \tag{S9}$$

where 0_n and r_n are vectors of size n and s_{n+1} is a real number. One also has

$$J_{n+1} = \begin{pmatrix} J_n & a_n e_n \\ a_n e_n^T & 1/2 \end{pmatrix}.$$

Then, ignoring the last row, we have

$$C^{n+1}_{-1} J_{n+1} = \begin{pmatrix} C^n_{-1} J_n & a_n s_n e_n \\ * & * \end{pmatrix},$$

and

$$J_{n+1}C_{-1}^{n+1} = \begin{pmatrix} J_nC_{-1}^n + a_ne_nr_n^T & a_ns_{n+1}e_n \\ * & * \end{pmatrix}.$$

From (S8) for $n+1$ we find

$$C_{-1}^{n+1}J_{n+1} - 2J_{n+1}C_{-1}^{n+1} = M_{n+1}J_{n+1}.$$

Ignoring the last row, this gives

$$C_{-1}^nJ_n - 2J_nC_{-1}^n - 2a_ne_nr_n^T = 0, \tag{S10}$$

and $a_ns_ne_n - 2a_ns_{n+1}e_n = 0$. The latter gives $2s_{n+1} = s_n$ and since $C_{-1}^1 = 1$ (and thus $s_1 = 1$) we find immediately that $s_{n+1} = 1/2^n$ (which was already given in [S3, S2]). So recursively we start with $C_{-1}^1 = 1$. If C_{-1}^n is known, then C_{-1}^{n+1} is given by (S9), where the last row (r_n^T, s_{n+1}) contains $s_{n+1} = 1/2^n$ and the vector r_n which is computed as the last row in

$$2a_ne_nr_n^T = C_{-1}^nJ_n - 2J_nC_{-1}^n.$$

The matrices C_1^n can be computed from C_{-1}^n by using $(C_{-1}^n)_{i,j} = (-1)^{i+j}(C_1^n)_{i,j}$. The latter relation, combined with (S3), gives

$$2\delta_{i_j} = (1+(-1)^{i+j})\sum_{k=1}^n (C_{-1}^n)_{i,k}(C_{-1}^n)_{j,k},$$

so that the even rows of C_{-1}^n are orthonormal vectors and the odd rows of C_{-1}^n are orthonormal. This is a useful check.

The matrix C_{-1}^{10} is given below. The matrices for $n < 10$ are found as the principal $n \times n$ submatrix of this matrix.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\sqrt{7}}{8} & \frac{\sqrt{21}}{8} & -\frac{\sqrt{35}}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{8} & \frac{3\sqrt{5}}{8} & -\frac{3\sqrt{7}}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{11}}{16} & -\frac{\sqrt{33}}{16} & -\frac{\sqrt{55}}{32} & \frac{3\sqrt{77}}{32} & -\frac{3\sqrt{11}}{32} & \frac{1}{32} & 0 & 0 & 0 & 0 \\
0 & -\frac{\sqrt{39}}{64} & -\frac{3\sqrt{65}}{64} & -\frac{\sqrt{91}}{16} & \frac{3\sqrt{13}}{16} & -\frac{\sqrt{143}}{64} & \frac{1}{64} & 0 & 0 & 0 \\
\frac{5\sqrt{15}}{128} & \frac{15\sqrt{5}}{128} & \frac{19\sqrt{3}}{128} & -\frac{\sqrt{105}}{128} & -\frac{25\sqrt{15}}{128} & \frac{5\sqrt{165}}{128} & -\frac{\sqrt{195}}{128} & \frac{1}{128} & 0 & 0 \\
0 & \frac{\sqrt{51}}{128} & \frac{3\sqrt{85}}{128} & \frac{5\sqrt{119}}{128} & \frac{9\sqrt{17}}{128} & -\frac{7\sqrt{187}}{128} & \frac{3\sqrt{221}}{128} & -\frac{\sqrt{255}}{256} & \frac{1}{256} & 0 \\
-\frac{7\sqrt{19}}{256} & -\frac{7\sqrt{57}}{256} & -\frac{3\sqrt{95}}{128} & -\frac{\sqrt{133}}{128} & \frac{9\sqrt{19}}{128} & \frac{5\sqrt{209}}{128} & -\frac{21\sqrt{247}}{512} & \frac{7\sqrt{285}}{512} & -\frac{\sqrt{323}}{512} & \frac{1}{512}
\end{pmatrix}$$

S1.2. Explicit expressions for D_1^n for small n. The multiwavelets for multiplicities $1 \le n \le 5$ are given in [S1, Table 1 on p. 259]. We have computed the matrices D_1^n and the matrices D_{-1}^n can be found using

$$(D_{-1}^n)_{i,j} = (-1)^{i+j+n-1}D_1^n.$$

All the matrices are of the form

$$D_1^n = \hat{D}_1^n \operatorname{diag}(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \dots, \frac{1}{\sqrt{2n-1}}),$$

where the diagonal matrix on the right contains the normalizing constants for the Legendre polynomials. This means that the multiwavelets of multiplicity n are given in terms of Legendre polynomials by

$$\begin{pmatrix} f_1^n(x) \\ f_2^n(x) \\ \vdots \\ f_n^n(x) \end{pmatrix} = \hat{D}_1^n \begin{pmatrix} P_0(2x-1) \\ P_1(2x-1) \\ \vdots \\ P_{n-1}(2x-1) \end{pmatrix}, \qquad x \in [0,1].$$

The fist ten matrices $\hat{D}_1^n$ are given by

$$\hat{D}_1^1 = \operatorname{diag}\left(\tfrac{\sqrt{2}}{2}\right)\left(1\right),$$

$$\hat{D}_1^2 = \operatorname{diag}\left(\tfrac{\sqrt{6}}{2} \quad \tfrac{3\sqrt{2}}{4}\right)\begin{pmatrix} 0 & 1 \\ -\frac{1}{3} & 1 \end{pmatrix},$$

$$\hat{D}_1^3 = \operatorname{diag}\left(\tfrac{5\sqrt{2}}{6} \quad \tfrac{5\sqrt{6}}{8} \quad \tfrac{\sqrt{10}}{3}\right)\begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & 1 \\ 0 & -\frac{1}{5} & 1 \\ \frac{1}{4} & -\frac{3}{4} & 1 \end{pmatrix},$$

$$\hat{D}_1^4 = \operatorname{diag}\left(\tfrac{7\sqrt{510}}{170} \quad \tfrac{\sqrt{42}}{4} \quad \tfrac{4\sqrt{1190}}{85} \quad \tfrac{\sqrt{210}}{16}\right)\begin{pmatrix} 0 & -\frac{2}{7} & \frac{10}{7} & 1 \\ \frac{2}{21} & -\frac{2}{7} & \frac{5}{21} & 1 \\ 0 & \frac{3}{32} & -\frac{15}{32} & 1 \\ -\frac{5}{21} & \frac{5}{7} & -\frac{23}{21} & 1 \end{pmatrix},$$

$$\hat{D}_1^5 = \operatorname{diag}\left(\tfrac{3\sqrt{186}}{62} \quad \tfrac{9\sqrt{38}}{38} \quad \tfrac{45\sqrt{514290}}{17143} \quad \tfrac{33\sqrt{798}}{608} \quad \tfrac{12\sqrt{1106}}{553}\right)$$

$$\begin{pmatrix} 2 & -6 & 5 & 21 & 9 \\ 0 & \frac{3}{2} & -\frac{15}{2} & 14 & 18 \\ -\frac{5}{2} & \frac{15}{2} & -\frac{67}{7} & -3 & \frac{270}{7} \\ 0 & -1 & 5 & -\frac{25}{2} & \frac{33}{2} \\ \frac{7}{4} & -\frac{21}{4} & \frac{235}{28} & -\frac{39}{4} & \frac{48}{7} \end{pmatrix},$$

$$\hat{D}_1^6 = \mathrm{diag}\left(\frac{11\sqrt{70}}{210}\quad \frac{55\sqrt{13490}}{5396}\quad \frac{506\sqrt{32270}}{48405}\quad \frac{845\sqrt{89034}}{129504}\quad \frac{4\sqrt{212982}}{1383}\quad \frac{13\sqrt{66}}{192}\right)$$
$$\begin{pmatrix} 0 & \frac{3}{11} & -\frac{15}{11} & \frac{28}{11} & \frac{36}{11} & 1 \\ -\frac{1}{11} & \frac{3}{11} & -\frac{19}{55} & -\frac{7}{55} & \frac{15}{11} & 1 \\ 0 & -\frac{39}{1012} & \frac{195}{1012} & -\frac{3647}{8096} & \frac{261}{736} & 1 \\ \frac{504}{9295} & -\frac{1512}{9295} & \frac{437}{1859} & -\frac{7}{55} & -\frac{3513}{9295} & 1 \\ 0 & \frac{3}{64} & -\frac{15}{64} & \frac{79}{128} & -\frac{135}{128} & 1 \\ -\frac{42}{143} & \frac{126}{143} & -\frac{205}{143} & \frac{259}{143} & -\frac{249}{143} & 1 \end{pmatrix},$$

$$\tilde{D}_1^7 = \mathrm{diag}\left(\frac{13\sqrt{54610}}{10922}\quad \frac{13\sqrt{30}}{80}\quad \frac{21931\sqrt{26922730}}{67306825}\quad \frac{299\sqrt{34230}}{26080}\quad \frac{129584\sqrt{790619170}}{1976547925}\quad \frac{221\sqrt{10758}}{20864}\quad \frac{32\sqrt{4169594}}{160369}\right)$$
$$\begin{pmatrix} -\frac{5}{13} & \frac{15}{13} & -\frac{19}{13} & -\frac{7}{13} & \frac{75}{13} & \frac{55}{13} & 1 \\ 0 & -\frac{1}{13} & \frac{5}{13} & -\frac{35}{39} & \frac{9}{13} & \frac{77}{39} & 1 \\ \frac{641}{12532} & -\frac{1923}{12532} & \frac{19235}{87724} & -\frac{99}{964} & -\frac{34539}{87724} & \frac{5445}{6748} & 1 \\ 0 & \frac{7}{299} & -\frac{35}{299} & \frac{265}{897} & -\frac{123}{299} & \frac{11}{897} & 1 \\ -\frac{15279}{296192} & \frac{45837}{296192} & -\frac{490215}{2073344} & \frac{62403}{296192} & \frac{40629}{518336} & -\frac{26895}{39872} & 1 \\ 0 & -\frac{9}{221} & \frac{45}{221} & -\frac{28}{51} & \frac{228}{221} & -\frac{899}{663} & 1 \\ \frac{363}{1024} & -\frac{1089}{1024} & \frac{3575}{2048} & -\frac{4697}{2048} & \frac{5091}{2048} & -\frac{4213}{2048} & 1 \end{pmatrix}$$

$$\tilde{D}_1^8 = \mathrm{diag}\left(\frac{\sqrt{7710}}{514}\quad \frac{21\sqrt{6890}}{2756}\quad \frac{744\sqrt{12310934790}}{58623499}\quad \frac{266441\sqrt{6383647010}}{10213835216}\quad \frac{51536\sqrt{56134851630}}{5613485163}\quad \frac{1729665\sqrt{530024297594}}{770944432864}\quad \frac{320\sqrt{541398}}{270699}\quad \frac{323\sqrt{37184290}}{6656768}\right)$$
$$\begin{pmatrix} 0 & -\frac{2}{5} & 2 & -\frac{14}{3} & \frac{18}{5} & \frac{154}{15} & \frac{26}{5} & 1 \\ \frac{2}{15} & -\frac{2}{5} & \frac{4}{7} & -\frac{4}{15} & -\frac{36}{35} & \frac{44}{21} & \frac{13}{5} & 1 \\ 0 & \frac{473}{14880} & -\frac{473}{2976} & \frac{223}{558} & -\frac{169}{310} & -\frac{2167}{89280} & \frac{37817}{29760} & 1 \\ -\frac{6778}{190315} & \frac{20334}{190315} & -\frac{7582}{47019} & \frac{75118}{570945} & \frac{128307}{1332205} & -\frac{394295}{799323} & \frac{230477}{570945} & 1 \\ 0 & -\frac{7007}{412288} & \frac{35035}{412288} & -\frac{367003}{1649152} & \frac{612339}{1649152} & -\frac{247467}{824576} & -\frac{246753}{824576} & 1 \\ \frac{30778}{576555} & -\frac{30778}{192185} & \frac{174373}{691866} & -\frac{133193}{494190} & \frac{1212121}{12684210} & \frac{1289921}{3459330} & -\frac{972907}{1001385} & 1 \\ 0 & \frac{1573}{40960} & -\frac{1573}{8192} & \frac{5369}{10240} & -\frac{10569}{10240} & \frac{63261}{40960} & -\frac{67977}{40960} & 1 \\ -\frac{143}{323} & \frac{429}{323} & -\frac{2123}{969} & \frac{2849}{969} & -\frac{12021}{3553} & \frac{3151}{969} & -\frac{25273}{10659} & 1 \end{pmatrix}$$

$$\tilde{D}_1^9 = \operatorname{diag}\Big(\tfrac{17\sqrt{3066}}{9198} \quad \tfrac{17\sqrt{305718}}{21837} \quad \tfrac{254099\sqrt{195086514}}{3218927481} \quad \tfrac{15181\sqrt{1350764030}}{294712152} \quad \tfrac{27011980\sqrt{4525963475306}}{24892799114183} \quad \tfrac{25517\sqrt{21593481910}}{1812040440} \quad \tfrac{488648\sqrt{9441081992706}}{1089355614543} \quad \tfrac{7429\sqrt{38399790}}{68743680} \quad \tfrac{64\sqrt{293331090}}{5176431}\Big)$$

$$\begin{pmatrix}
\frac{14}{17} & -\frac{42}{17} & \frac{60}{17} & -\frac{28}{17} & -\frac{108}{17} & \frac{220}{17} & \frac{273}{17} & \frac{105}{17} & 1\\
0 & \frac{7}{68} & -\frac{35}{68} & \frac{22}{17} & -\frac{30}{17} & -\frac{11}{136} & \frac{559}{136} & \frac{55}{17} & 1\\
-\frac{16093}{254099} & \frac{48279}{254099} & -\frac{8565}{29894} & \frac{118027}{508198} & \frac{179253}{1016396} & -\frac{893519}{1016396} & \frac{174447}{254099} & \frac{26040}{14947} & 1\\
0 & -\frac{257}{15181} & \frac{1285}{15181} & -\frac{147357}{667964} & \frac{240645}{667964} & -\frac{7915}{30362} & -\frac{117611}{333982} & \frac{132641}{166991} & 1\\
\frac{3125551}{108047920} & -\frac{9376653}{108047920} & \frac{23338523}{172876672} & -\frac{116724881}{864383360} & \frac{12646833}{864383360} & \frac{228799241}{864383360} & -\frac{53297647}{108047920} & \frac{83601}{1271152} & 1\\
0 & \frac{5687}{408272} & -\frac{28435}{408272} & \frac{209097}{1122748} & -\frac{380505}{1122748} & \frac{2103635}{5307536} & -\frac{498977}{4490992} & -\frac{2187635}{3648931} & 1\\
-\frac{915629}{15636736} & \frac{2746887}{15636736} & -\frac{8797695}{31273472} & \frac{10308641}{31273472} & -\frac{7081371}{31273472} & -\frac{4364723}{31273472} & \frac{5933817}{7818368} & -\frac{584115}{459904} & 1\\
0 & -\frac{286}{7429} & \frac{1430}{7429} & -\frac{3934}{7429} & \frac{7950}{7429} & -\frac{165350}{96577} & \frac{16006}{7429} & -\frac{189745}{96577} & 1\\
\frac{9295}{16384} & -\frac{27885}{16384} & \frac{46137}{16384} & -\frac{62699}{16384} & \frac{74655}{16384} & -\frac{77605}{16384} & \frac{67353}{16384} & -\frac{43971}{16384} & 1
\end{pmatrix}$$

$$\tilde{D}_1^{10} = \operatorname{diag}\Big(\tfrac{19\sqrt{195734}}{139810} \quad \tfrac{57\sqrt{10130}}{20260} \quad \tfrac{60154\sqrt{193114380030}}{32185730005} \quad \tfrac{2685897\sqrt{173245498730}}{692981994920} \quad \tfrac{32615476\sqrt{1772129191206}}{19198066238065} \quad \tfrac{320784657\sqrt{49082496344587186}}{30204613135130576} \quad \tfrac{41893024\sqrt{8789678062}}{2113917573911} \quad \tfrac{8339225\sqrt{11572967547441690}}{799643641604608} \quad \tfrac{64\sqrt{2863718}}{214489} \quad \tfrac{115\sqrt{356469906}}{14488576}\Big)$$

$$\begin{pmatrix}
0 & \frac{14}{19} & -\frac{70}{19} & \frac{176}{19} & -\frac{240}{19} & -\frac{11}{19} & \frac{559}{19} & \frac{440}{19} & \frac{136}{19} & 1\\
-\frac{14}{57} & \frac{14}{19} & -\frac{10}{9} & \frac{154}{171} & \frac{13}{19} & -\frac{583}{171} & \frac{455}{171} & \frac{385}{57} & \frac{221}{57} & 1\\
0 & -\frac{3419}{90231} & \frac{17095}{90231} & -\frac{1425347}{2887392} & \frac{774985}{962464} & -\frac{11}{19} & -\frac{23868}{30077} & \frac{1267475}{721848} & \frac{28101}{12664} & 1\\
\frac{33352}{895299} & -\frac{33352}{298433} & \frac{1864805}{10743588} & -\frac{1847363}{10743588} & \frac{44417}{3581196} & \frac{3764651}{10743588} & -\frac{1676545}{2685897} & \frac{6615}{298433} & \frac{1060307}{895299} & 1\\
0 & \frac{16997513}{1565542848} & -\frac{84987565}{1565542848} & \frac{452164559}{3131085696} & -\frac{268652845}{1043695232} & \frac{72951373}{260923808} & -\frac{4507737}{260923808} & -\frac{390871835}{782771424} & \frac{5774883}{13732832} & 1\\
-\frac{16858127}{641569314} & \frac{16858127}{213856438} & -\frac{480980555}{3849415884} & \frac{23247763}{167365908} & -\frac{387592627}{5560267388} & -\frac{6519604355}{50042406492} & \frac{383279392}{962353971} & -\frac{1713869980}{4170200541} & -\frac{1017012947}{4170200541} & 1\\
0 & -\frac{8340319}{670288384} & \frac{41701595}{670288384} & -\frac{903622335}{5362307072} & \frac{1727236023}{5362307072} & -\frac{74014413}{167572096} & \frac{28755857}{83786048} & \frac{1728420455}{10724614144} & -\frac{505997741}{564453376} & 1\\
\frac{1274416}{19013433} & -\frac{1274416}{6337811} & \frac{51583246}{158445275} & -\frac{191156966}{475335825} & \frac{144242082}{411957715} & -\frac{14844610}{247174629} & -\frac{83114491}{158445275} & \frac{2550408241}{2059788575} & -\frac{9702647507}{6179365725} & 1\\
0 & \frac{663}{16384} & -\frac{3315}{16384} & \frac{9163}{16384} & -\frac{18819}{16384} & \frac{407099}{212992} & -\frac{42891}{16384} & \frac{1218885}{425984} & -\frac{74409}{32768} & 1\\
-\frac{4862}{6555} & \frac{4862}{2185} & -\frac{1612}{437} & \frac{33124}{6555} & -\frac{13428}{2185} & \frac{43868}{6555} & -\frac{13951}{2185} & \frac{2213}{437} & -\frac{19637}{6555} & 1
\end{pmatrix}$$

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